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M.Sc. Sem-II, Paper VI (Complex Analysis)

Power series

Some Definitions

(1) A subset $G \subseteq$ is open if, for each $z \in S$, there is an $\varepsilon > 0$ such that $B(z;\varepsilon) \subseteq G$. The point z_0 is said to be an interior point of the set $S \subseteq$ if there exists an $\varepsilon > 0$ such that $B(z,\varepsilon) \subseteq S$. Further interior of S, written int S, is the set $\bar{S} = \cap\{G : G$ is open and $G \subseteq S\}$.
The closure of $S \subseteq$, denoted by $\bar{S}$, is the set $\bar{S} = \cap\{F : F$ is closed and $F \supseteq S\}$. The boundary of S, denoted by ∂S and defined by $\partial S = \bar{S} \cap (\overline{X - S})$. Further, a subset S is dense if $\bar{S} =$.

(2) A metric space (X, d) is connected if the only subsets of X which are both open and closed are X and the empty set. Further, a subset $S \subseteq X$ is connected if the metric space (S, d) is connected.

(3) If G is an open set in a and $f : G \to$, then f is differentiable at a point a in G if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ exists. It is denoted by $f'(a)$ and called derivative of f at a.

(4) If f is defferentiable on G, then we define $f' : G \to$. If f' is continuous then we say that f is continuously differentiable.

(5) ~~[illegible]~~ A differentiable function such that each successive derivative is again differentiable is called infinitely differentiable.

(6) A function $f : G \to$ is analytic if f is continuously differentiable on G.

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Power Series:- A series of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$ where $a_1, a_2, ---, a_n, ---$ are constants, is called power series about a.

Example. The geometrical series $\sum_{n=0}^{\infty} z^n$ is power series about 0 and for $|z|<1$, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

Theorem:- For given power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ define a number $0 \leq R \leq \infty$ by $\frac{1}{R} = \text{Lt} \sup |a_n|^{\frac{1}{n}}$, then

(a) If $|z-a|<R$, the series converges absolutely.

(b) If $|z-a|>R$, the term of series become unbounded and so series diverges.

(c) If $0<r<R$, then the series converges uniformly on $\{z : |z-a| \leq r\}$.

moreover the number R is the only number having properties (a) and (b).

Proof:- we may suppose that $a=0$.

(a) If $|z|<R$, then there is an r such that $|z|<r<R$, $(\frac{1}{r} > \frac{1}{R})$. Thus by definition of limit sup, there is an integer N such that

$$|a_n|^{1/n} < \frac{1}{r} \quad \text{for all } n \geq N.$$

$$|a_n| < \frac{1}{r^n} \quad \text{for all } n \geq N.$$

$$|a_n z^n| < \left(\frac{|z|}{r}\right)^n \quad \text{for all } n \geq N.$$

Thus the series $\sum_{n=0}^{\infty} a_n z^n$ is dominated by the series $\sum \left(\frac{|z|}{r}\right)^n$. Since the geometric series $\sum_{n=0}^{\infty} \left(\frac{|z|}{r}\right)^n$ converges for $|z|<r$, the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for each $|z|<R$.

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